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Shape of a small universe: Signatures in the cosmic microwave background

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We consider the most general parametrization of flat topologically compact universes, complementing the work of Scannapieco, Levin and Silk to include nontrivial shapes. We find that modifications in the shape of the fundamental domain will lead to distinct signatures in the anisotropy of the cosmic microwave radiation. We make a preliminary assessment of the effect on three statistics: the angular power spectrum, the distribution of identified "circles" on the surface of last scattering and the correlation function of antipodal points.

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With the dramatic improvement of cosmological observations in the past few years, efforts are underway to precisely characterize the nature and structure of space-time. One property which has generated much interest is the possibility that we live on a manifold which has a compact, nontrivial topology [1]. A multitude of tests and observables have been proposed which would allow us to pin down the topological class of the universe we live in. Under the assumption that we live in a smooth Friedmann-Robertson-Walker universe with perturbations set up adiabatically in the early universe, measurements of the cosmic microwave background (CMB) anisotropy seem to indicate that our universe is flat [2]. This greatly reduces the set of models one needs to look at and simplifies the boundary conditions of the wave functions one needs to consider. Scannapieco, Levin and Silk [3] have identified complete sets of functions for each topology for the case where the identification vectors are orthogonal.

In the context of Kaluza-Klein theories, it has recently been pointed out in [4,5] that nonorthogonality of the identification vectors will lead to interesting effects in the mass differences of Kaluza-Klein states. It was shown in [4] that the mass gap of the graviton will become dependent on what Dienes calls the shape parameter, i.e., the angle between identification vectors. While he considered the case of a compact 2-dimensional space rather than the standard 3-dimensional space we live in, we shall see what happens if we apply such a transformation to a compactified version of our 3-dimensional space in the case of no extra dimensions.

Without loss of generality let us consider the simplest, flat, nontrivial topology—the hypertorus in the classification of [6]. This will allow us to outline the formalism which can be used in the general case. Consider a 3-manifold with coordinates $\vec{\xi}=(\xi_1,\xi_2,\xi_3)$ on which any function $f(\xi_1,\xi_2,\xi_3)$ satisfies the following property:

$$f(\xi_1+2\pi,\xi_2,\xi_3)=f(\xi_1,\xi_2+2\pi,\xi_3)=f(\xi_1,\xi_2,\xi_3+2\pi)=f(\xi_1,\xi_2,\xi_3). \quad (1)$$

We can think of this manifold as tiled by cubes with sides of length 2π. Alternatively we can think of it as simply consisting of a cube with opposite faces identified; for convenience let us call it a *fundamental domain*. We can now rescale each direction in such a way as to have different identification scales in each direction by defining a matrix $\mathbf{L}$:

$$\mathbf{L}=\mathrm{diag}(R_1,R_2,R_3) \quad (2)$$

and a new set of coordinates: $\vec{\chi}=\mathbf{L}\cdot\vec{\xi}$. It will be useful to divide this transformation into two: one that is simply the product of the identity by some overall scale factor R and another transformation which is diagonal with unit determinant. The first transformation corresponds to an overall volume transformation, modifying the volume of the fundamental domain by a factor of R^3. The second transformation induces anisotropic scale differences in the sides of the fundamental domain (changing it into a parallelipiped). A coordinate invariant definition orders the eigenvalues of this second transformation, $\lambda_1>\lambda_2>\lambda_3$ and defines $\sigma_1=\lambda_2/\lambda_1$ and $\sigma_2=\lambda_3/\lambda_1$.

We now wish to consider an additional linear transform that changes the directions of the identification vectors. Naturally we are not interested in overall rotations of all vectors, given that ultimately any observable we construct should be rotationally invariant. Consider a set of 3 vectors: $\vec{s}_1$, $\vec{s}_2$, $\vec{s}_3$. Given that we wish to factor out overall rotations, we can align $\vec{s}_1$ with ξ_1, have $\vec{s}_2$ lie on the (ξ_1,ξ_2) plane and $\vec{s}_3$ will have an arbitrary direction. We can define a shape transformation matrix, $\mathbf{S}$, such that its three columns are the three vectors, $\vec{s}_i$. A convenient parametrization is the following:

$$\mathbf{S}=\omega\begin{pmatrix} 1 & \cos\alpha_1 & \cos\alpha_2 \\ 0 & \sin\alpha_1 & \sin\alpha_2\cos\alpha_3 \\ 0 & 0 & \sin\alpha_2\sin\alpha_3 \end{pmatrix}. \quad (3)$$

Note that the prefactor, $\omega=(\sin\alpha_1\sin\alpha_2\sin\alpha_3)^{-1/3}$, ensures that $\det\mathbf{S}=1$. This parametrization is such that $\vec{s}_1\cdot\vec{s}_2=\omega^2\cos\alpha_1$, $\vec{s}_1\cdot\vec{s}_3=\omega^2\cos\alpha_2$ and $\vec{s}_2\cdot\vec{s}_3=\omega^2(\cos\alpha_1\cos\alpha_2+\sin\alpha_1\sin\alpha_2\cos\alpha_3)$. The geometrical interpretation of this transformation is simple: we are changing the angles of the corner of our fundamental domain while preserving its volume. We can now consider a new set of coordinates $\vec{x}=\mathbf{SL}\cdot\vec{\xi}$. The identification vectors are not orthogonal anymore.

Note that we are allowed to perform these transformations without jeopardizing the suitability of the manifold as a cosmological model. We start off with a locally homogeneous and isotropic space-time with nontrivial identifications that are permitted. We then simply apply a linear transformation which preserves the homogeneity and isotropy of the manifold as well as differentiability. Indeed if we consider a generalization to a flat manifold in N dimensions with analogous topological constraints, we are allowed to perform any nonsingular linear transformation and the resulting manifold will still have the same differentiability class. Such a transformation will be characterized in terms of N rescaling parameters and $N(N-1)/2$ shape parameters. Furthermore, these transformations clearly cannot change the topology class of the manifold. In the example we are considering, the topology remains that of a hypertorus and the geometry remains the same: zero curvature.

The plane wave solutions to the Laplaces equations will be modified. A complete set of orthogonal solutions is given by $\psi_{\vec{n}}(\vec{\xi})=\exp(-i\vec{n}\cdot\vec{\xi})$ with $\vec{n}=2\pi(n_1,n_2,n_3)$. This can be mapped directly onto a complete set of orthogonal solutions on the transformed space: $\phi_{\vec{k}}(\vec{x})\equiv\exp(-i\vec{k}\cdot\vec{x})=\psi_{\vec{n}}(\vec{\xi})$ where the last equality implies $\vec{k}=\mathbf{L}^{-1}\mathbf{S}^{-1}\mathbf{n}$. As above the wave numbers are quantized, but the effect of $\mathbf{S}$ is to mix up the quantization constraints through the shape angles. The explicit form of the wave number for this topology is

$$\frac{\omega R_1 k_1}{2\pi}=n_1-n_2\cot\alpha_1+n_3\left(\cot\alpha_1\cot\alpha_3-\frac{\cot\alpha_2}{\sin\alpha_3}\right)$$

$$\frac{\omega R_2 k_2}{2\pi}=n_2\frac{1}{\sin\alpha_1}-n_3\frac{\cot\alpha_3}{\sin\alpha_1} \tag{4}$$

$$\frac{\omega R_3 k_3}{2\pi}=n_3\frac{1}{\sin\alpha_2\sin\alpha_3}.$$

In this note we wish to focus on the effect of shape. We will restrict ourselves to $R_1=R_2=R_3=R$ and two simple, one parameter families of shape transformations: (a) We fix $\alpha_2=\alpha_3=\pi/2$ and vary $\alpha_1=\alpha\in[0,\pi/2]$. This essentially corresponds to collapsing one of the identification vectors of the fundamental domain on the plane defined by the other two. (b) We fix $\hat{s}_1\cdot\hat{s}_2=\hat{s}_2\cdot\hat{s}_3=\hat{s}_3\cdot\hat{s}_1=\cos\alpha$ and vary $\alpha\in[0,\pi/2]$.

An hypothesis was put forward in [3] that the minimum traversable distance of the space defines the scale at which observable effects appear in statistics of the CMB such as the angular power spectrum or the geometry and frequency of ghost images. Although our shape transformation has unit determinant, the minimum traversable distance (d_{min}) may be greater or less than the one in the cubic domain. For the two families above we find that

$$d_{min}=R\omega\begin{cases}1 & \text{for } \cos\alpha\in[0,0.5],\\ \sqrt{2(1-\cos\alpha)} & \text{for } \cos\alpha\in[0.5,1],\end{cases} \tag{5}$$

and in Fig. 1 we plot this quantity for the two families we are considering. We find that, indeed, d_{min} does vary as a function of shape parameter. Although there is a slight increase for $\cos\alpha<0.5$, both parametrizations A and B lead to a dramatic decrease in d_{min} for $\cos\alpha\sim1$.

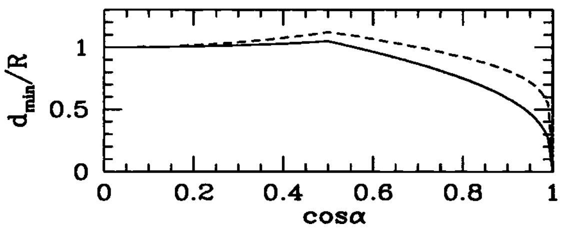

FIG. 1. The minimum traversible distance for the two families of shape transformations that we are considering as a function of the cosine of the angle. The solid (dashed) line corresponds to type A(B).

The question we now wish to address is how will this be refelected in some of the statistics that have been proposed to quantify topology from the CMB.

Let us first focus on the angular power spectrum of the CMB. In a flat universe, the anisotropy in the CMB, $\Delta T(\hat{\theta})$ (where $\hat{\theta}$ is the unit vector pointing in a given direction in the sky), is dominated by the Sachs-Wolfe term, i.e.,

$$\Delta T(\hat{\theta})=-\frac{1}{3}\Phi(\eta_*,\hat{\theta}d_*), \tag{6}$$

where $\Phi(\eta,\mathbf{x})$ is the gravitational potential, η_* is the time of last scattering, and d_* is the comoving distance to the surface of last scattering. We will be working in Fourier space where $f(\mathbf{x})=\Sigma_{\vec{k}}f(\vec{k})\exp(-i\vec{k}\cdot\vec{x})$. Assuming statistical homogeneity and isotropy we have that $\langle\Phi^*(\eta_0,\vec{k})\Phi^*(\eta_0,\vec{k}')\rangle=\mathcal{P}(k)\delta_{\vec{k},\vec{k}'}$. Defining $a_{\ell m}=\int d\hat{\theta}T(\hat{\theta})Y^*_{\ell m}(\hat{\theta})$ we have that

$$\langle a^*_{\ell m}a_{\ell' m'}\rangle=(4\pi)^2 i^{l'-l}\sum_{\vec{k}}\mathcal{P}(k)j_\ell(kd_*)j_{\ell'}(kd_*)\times Y_{\ell m}(\hat{k})Y^*_{\ell' m'}(\hat{k}). \tag{7}$$

In general, for arbitrary topology the covariance matrix of the $a_{\ell m}$ will not be diagonal. The fact that there are preferred orientations in space (defined by the identification vectors) will induce correlations in the a_{lm} between different ℓ and m, which depend on the orientation of the $Y_{\ell m}(\hat{\theta})$ basis. In this letter, as in [3], we wish to look at the overall impact on a rotationally invariant measure of the angular power spectrum. We shall consider

$$C_\ell\equiv\frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell}|a_{\ell m}|^2=(4\pi)\sum_{\vec{k}}\mathcal{P}(k)j^2_\ell(kd_*). \tag{8}$$

For the infinite volume case this converges to the standard angular power spectrum.

In Fig. 2 we plot angular power spectra for $\alpha=\pi/2$, $\pi/4$, and $\pi/8$. We are considering a universe with an identification scale $R=d_*$. For the cubic domain the features identified by

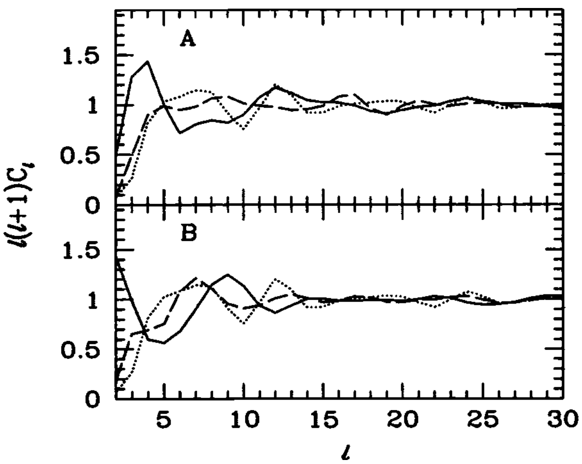

FIG. 2. The angular power spectrum for a selection of three values of the shape parameter for the two families of transformations, *A* and *B*. The angle α takes values $\pi/2$ (dotted), $\pi/4$ (long-dashed) and $\pi/8$ (solid).

[3] are manifest, i.e., one finds a supression of power for low ℓ and a set of oscillatory features out to large ℓ. Changing the value of α will shift the oscillatory features around but can have a curious effect at the lowest multipoles: the solid lines (corresponding to $\alpha=\pi/8$) show an increase in power due to the fact that the shape transformation can lead to the regeneration of small wave number modes in the integral. Indeed we find this to be a general trend [10].

All real space statistics rely, to some extent, on the repetition of patterns on the surface of last scattering. The surface of last scattering will intersect points in space which are identified. This leads to perfect correlations between certain points (or pixels) of a map of the CMB. The more points which are identified, the more significant a detection of topology will be. We can attempt to quantify the effect of shape by estimating how many points (or structure, such as, for example, circles) will be identified for a given value of α in either of the transformations *A* and *B*. Two points $\vec{x}$ and $\vec{x}'$ lie on the surface of last scattering if they satisfy $|x|^2=|x'|^2=d_*^2$ and can be identified with one another if there exist a trio of integers, (n_1,n_2,n_3) such that $\vec{x}'=\vec{x}+n_1\vec{s}_1+n_2\vec{s}_2+n_2\vec{s}_2$. Combining these two sets of equations one finds that a given trio of integers satisfies this condition if

$$(n_1^2+n_2^2+n_3^2+2n_1n_2\hat{s}_1\cdot\hat{s}_2+2n_2n_3\hat{s}_2\cdot\hat{s}_3+2n_3n_1\hat{s}_3\cdot\hat{s}_1)^{1/2}/[\hat{s}_1\cdot(\hat{s}_2\times\hat{s}_3)]^{1/3}<\frac{2d_*}{R}. \quad (9)$$

For example, for an orthogonal trio of $\hat{s}_i$ and $R=2d_*$ one has 3 pairs of identified structures. This condition for transformation *A* is

$$(n_1^2+n_2^2+n_3^2+2n_1n_2\cos\alpha)/(\sin\alpha)^{1/3}<\frac{2d_*}{R} \quad (10)$$

and for transformation *B* it is

$$[n_1^2+n_2^2+n_3^2+2\cos\alpha(n_1n_2+n_2n_3+n_3n_1)]^{1/2}/(1-\cos\alpha)\times\sqrt{(1+2\cos\alpha)^{1/3}}<\frac{2d_*}{R}. \quad (11)$$

As $\cos\alpha\rightarrow1$, more roots are possible, leading to stronger correlations between points on the surface of last scattering.

There is in fact a simple way of understanding how the number of roots (or matched structures) increases with $\cos\alpha$. Let us first consider transformation *A*. The direction of the minimum traversable distance is given (with $\cos\alpha>0.5$) by $\vec{s}_1-\vec{s}_2$. As we increase $\cos\alpha$, we decrease d_{min} so that the number of identified structures will be given by the integer part of $2d_*/d_{min}$. Given that $\lim_{\cos\alpha\rightarrow1}d_{min}=0$, there will be more matched structures with increasing $\cos\alpha$. For transformation *B*, the reasoning is the same except there are now three possible directions: $\vec{s}_1-\vec{s}_2$, $\vec{s}_2-\vec{s}_3$ and $\vec{s}_3-\vec{s}_1$. We then have that the number of matched structures is given by the integer part of $6d_*/d_{min}$.

Let us now focus on two real space statistics that have been advocated for identifying topology in the CMB: matching circles in the sky by [7] and the correlation function of antipodal points of [8]. In [7] Cornish, Spergel, and Starkman have proposed matching the circles which arise from the intersection of the surface of last scattering with the identified boundaries of the fundamental domain. One way of thinking about this is to consider a given identification vector $\vec{s}$. One can then construct two perpendicular planes at each end and find the lines of their intersection with the sphere of radius d_* and center at the midpoint of the vector. For example, for a cubic domain of size $R<2d_*$ one expects at least 3 pairs of circles with centers on the 3 axes of identification. Pairs of circles will intersect (with each other) if $R<\sqrt{2}d_*$, and each circle will subtend an angle $\theta_0=\pi/2-\arcsin(R/2d_*)$ on the sky. For a cubic domain only pairs of big circles will intersect. Let us now consider a fundamental domain with a nontrivial shape parameter. The identification vectors will not be orthogonal and it will be possible to have pairs of small circles that intersect. The condition is that $\alpha<\theta_0$. This is then a signature which could distinguish between cubic and non-cubic domains. More generally, if one identifies more than one pair of circles and the vectors joining their centers are not perpendicular, then there is evidence for nontrivial shape. Naturally one would have to distinguish it from the hexagonal fundamental domains, where the identification vectors are at 60°.

Levin *et al.* in [8] have suggested that a good statistic for identifying nontrivial topology is the correlation of antipodal points on the surface of last scattering:

$$C_A(\hat{\theta})\equiv\langle\Delta T(-\hat{\theta})\Delta T(\hat{\theta})\rangle\propto\sum_{\mathbf{k}}\mathcal{P}(k)\cos(2d_*\mathbf{k}\cdot\hat{\theta}). \quad (12)$$

The patterns which appear will reflect the geometry of the fundamental domains. Again one might expect that a non-cubic domain may lead to a different signature; for example it has been shown in [8] that a hexagonal fundamental do-

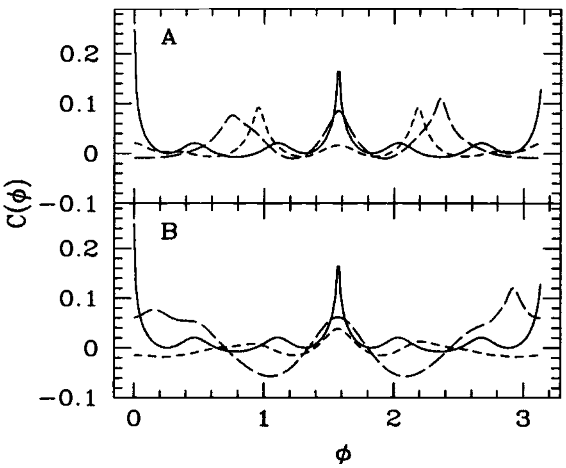

FIG. 3. A one-dimensional slice of the antipodal correlation function for a selection of three values of the shape parameter for the two families of transformations, *A* and *B*. The angle α takes values $\pi/2$ (solid), $\pi/4$ (short-dash) and $\pi/8$ (long-dash).

main will have a very different signature than a cubic one. As an illustration of this statistic for nontrivial shapes we have calculated $C_A(\hat{\theta})$ for the one-dimensional slice $\hat{\theta} = (\cos\phi, \sin\phi, 0)$. We can clearly see the differences in Fig. 3 where we plot $C_A(\phi)$ for the two families of shape transformations.

In this paper we have taken a first look at how the shape of the fundamental domain of a flat, topologically nontrivial universe might affect a few different statistics of the CMB. There clearly seem to be observable signatures which will constrain the values of the shape parameters. We will now finish with a few comments. To begin with, it is clear that when it is claimed that one can use the cosmic microwave background to constrain the topology of the universe, it is in fact meant that one is constraining certain geometric properties which arise when the universe is topologically compact. For example, if the identification scales are infinite, then the shape transformations are irrelevant. When choosing the topology of the universe (from the six possibilities discussed in [3]), one must also include the 6 parameters which describe the geometry and scale of the fundamental domain. One can then proceed to apply the variety of statistics that have been advocated. It is possible, however, to incorporate the parameters describing the fundamental domain into the standard likelihood methods which have been used to constrain cosmological parameters. The standard parameters p_i for Gaussian models should be extended to include another six parameters $(R, \sigma_1, \sigma_2, \alpha_1, \alpha_2, \alpha_3)$ for topology.

Topologically nontrivial universes are no longer statistically isotropic. One must choose an overall orientation of the fundamental domain. This complicates considerably the evaluation of the likelihood function for a given topology primarily because one cannot use the radical compression methods [9]. One must work with the full covariance matrix in real space, $C(\hat{\theta}, \hat{\theta})$, which is no longer simply a function of $\hat{\theta}\cdot\hat{\theta}$. This is a daunting computational task: not only is one extending parameter space by another six factors, but one must also marginalize over all possible orientations in space. Furthermore, what was once a one-dimensional radial integral over wave number becomes a three-dimensional integral which only picks up contributions from the selected modes on the Fourier grid. If one is to correctly constrain the properties of the fundamental domain of our universe, it is essential to develop novel techniques which can incorporate these various subtleties into likelihood methods. One should point out, however, the impressive increase in speed of the algorithms for performing the various steps in maximum likelihood estimation. Hopefully this means that the challenge of constraining topologically nontrivial universes will be met and that it will be possible to do so with the data which will come from the MAP satellite.

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